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BIFURCATION OF SOLUTIONS OF STATICS PROBLEMS OF THE NON-LINEAR THEORY OF ELASTICITY*

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The problem of the bifurcation of the equilibrium mode of an elastic body is considered on the basis of the three-dimensional equations of the non-linear theory of elasticity. Necessary and sufficient conditions for the solvability of an inhomogeneous linearized boundary value problem are derived for conservative external forces and are utilized to formulate the bifurcation equation. In application to three-dimensional problems of the non-linear theory of elasticity, a procedure is constructed for the Lyapunov-Schmidt method in operator form which enables a number of solutions to be determined that branch off at the bifurcation point and enable asymptotic representations to be obtained for solutions under the almost critical loads. The general theory is illustrated by investigation of the post-critical behaviour of a thick-walled cylinder loaded on the side surface.

1. The equilibrium of an elastic body is described by the equations /1/

$$\nabla\cdot\mathbf{D}+\mathbf{k}=0,\quad \mathbf{D}(\mathbf{r},\mathbf{C}(\mathbf{r}))=dW/d\mathbf{C} \qquad (1.1)$$
$$\mathbf{C}=\nabla\mathbf{R},\quad \mathbf{r}=x_k\mathbf{i}_k,\quad \nabla=\mathbf{i}_k\partial/\partial x_k$$

Here $\mathbf{D}$ is the Piola stress tensor, W is the specific strain potential energy, $\mathbf{C}$ is the spatial gradient, ∇ is the nabla operator in the reference (undeformed) configuration, x_k are Cartesian coordinates of the undeformed body, $\mathbf{i}_k$ are the Cartesian coordinate directions, $\mathbf{R}$ is the position vector of the position of a point of the deformed body, and $\mathbf{k}$ is the volume force intensity per unit volume of the reference configuration. The explicit dependence of the Piola tensor on $\mathbf{r}$, i.e., on the Lagrange coordinates in (1.1), holds for an inhomogeneous body. For a homogeneous body $\mathbf{D}=\mathbf{D}(\mathbf{C}(\mathbf{r}))$.

We will assume that the body surface σ with unit external normal $\mathbf{n}$ in the reference configuration consists of three parts. The external load $\mathbf{n}\cdot\mathbf{D}=\mathbf{f}$ is given on σ_1, the rotation vector $\mathbf{R}=\mathbf{R}_*$ on σ_2, and particles of the surface σ_3 in the deformed state made frictionless contact with the smooth solid surface Π.

The external force intensities $\mathbf{k}$ and $\mathbf{f}$ are not necessarily given functions of the Lagrange coordinates. They can depend in a given manner on the vector $\mathbf{R}$ and the spatial gradient $\mathbf{C}$. This dependence is later assumed such that the load is conservative. This means that the elementary work of the external load is a variation of a certain functional Φ

$$\int_v \mathbf{k}[\mathbf{r},\mathbf{R}(\mathbf{r}),\mathbf{C}(\mathbf{r})]\cdot\delta\mathbf{R}\,dv+\int_{\sigma_1}\mathbf{f}[\mathbf{r},\mathbf{R}(\mathbf{r}),\mathbf{C}(\mathbf{r})]\cdot\delta\mathbf{R}\,d\sigma=\delta\Phi \qquad (1.2)$$

where v is the volume occupied by the elastic body in the reference configuration.

Let $\mathbf{R}=\boldsymbol{\rho}(\mathbf{r})$ be a certain known solution of the equilibrium problem that governs the subcritical state of the elastic body. We set $\mathbf{R}=\boldsymbol{\rho}+\mathbf{w}$ to study the solutions close to $\boldsymbol{\rho}$. Using (1.1), we write the non-linear boundary value problem in the vector $\mathbf{w}$ as follows

$$\nabla\cdot\mathbf{D}^{\cdot}+\mathbf{k}^{\cdot}=-\mathbf{K}\ \text{in}\, v \qquad (1.3)$$

$$\mathbf{n}\cdot\mathbf{D}^{\cdot}-\mathbf{f}^{\cdot}=\mathbf{F}\ \text{on}\,\sigma_1,\quad \mathbf{w}=0\ \text{on}\,\sigma_2 \qquad (1.4)$$

$$\mathbf{n}\cdot\mathbf{D}^{\cdot}\cdot\mathbf{G}+S\mathbf{B}\cdot\mathbf{w}=\boldsymbol{\tau},\quad \mathbf{N}\cdot\mathbf{w}=\varphi\ \text{on}\,\sigma_3 \qquad (1.5)$$
$$\boldsymbol{\tau}\cdot\mathbf{N}=0,\quad \mathbf{D}^{\cdot}=\frac{d}{d\eta}\mathbf{D}(\mathbf{r},\nabla\boldsymbol{\rho}+\eta\nabla\mathbf{w})|_{\eta=0}$$
$$\mathbf{f}^{\cdot}=\frac{d}{d\eta}\mathbf{f}(\mathbf{r},\boldsymbol{\rho}+\eta\mathbf{w},\nabla\boldsymbol{\rho}+\eta\nabla\mathbf{w})|_{\eta=0}$$

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$$\mathbf{k}^{\cdot} = \frac{d}{d\eta}\mathbf{k}(\mathbf{r}, \boldsymbol{\rho} + \eta\mathbf{w}, \nabla\boldsymbol{\rho} + \eta\nabla\mathbf{w})|_{\eta=0}$$

$$\mathbf{G} = \mathbf{E} - \mathbf{NN},\ \mathbf{B} = -\mathbf{G}\cdot(\nabla\boldsymbol{\rho})^{-1}\cdot\nabla\mathbf{N}$$

$$S = \mathbf{N}\cdot\mathbf{T}\cdot\mathbf{N}\det(\nabla\boldsymbol{\rho})(\mathbf{n}\cdot\Lambda^{-1}\cdot\mathbf{n})^{1/2},\ \Lambda = \nabla\boldsymbol{\rho}\cdot(\nabla\boldsymbol{\rho})^T$$

Here $\mathbf{N}$ is the normal to the body surface in the deformed state that corresponds to the fundamental solution $\boldsymbol{\rho}$, $\mathbf{B}$ is the second fundamental tensor /2/ of the surface Π, $\mathbf{E}$ is the unit tensor, and $\mathbf{T}$ is the Cauchy stress tensor in the fundamental (subcritical) state. The difference expressions $\mathbf{D}^{\cdot}, \mathbf{k}^{\cdot}, \mathbf{f}^{\cdot}$ in (1.3)-(1.5) are linear in $\mathbf{w}$. The expressions $\mathbf{K}, \mathbf{F}, \boldsymbol{\tau}, \varphi$ do not contain linear terms in $\mathbf{w}$. This means that if we set $\mathbf{w} = \varepsilon\mathbf{w}_0$, the expansion of these expressions in powers of the parameter ε will start with terms of not less than the second order.

The linearized boundary conditions obtained in /3/ on the contact surface of a pre-stressed elastic body with an absolutely rigid body are utilized in (1.5).

The fundamental solution $\boldsymbol{\rho}$ depends on the loading parameter p. Therefore, the parameter p takes part in the formulation of the boundary value problem (1.3)-(1.5).

Discarding non-linear terms in (1.3)-(1.5), we obtain a homogeneous linear boundary value problem

$$\nabla\cdot\mathbf{D}^{\cdot} + \mathbf{k}^{\cdot} = 0 \text{ in } v \tag{1.6}$$

$$\mathbf{n}\cdot\mathbf{D}^{\cdot} - \mathbf{f}^{\cdot} = 0 \text{ on } \sigma_1,\ \mathbf{w} = 0 \text{ on } \sigma_2 \tag{1.7}$$

$$\mathbf{n}\cdot\mathbf{D}^{\cdot}\cdot\mathbf{G} + S\mathbf{B}\cdot\mathbf{w} = 0,\quad \mathbf{N}\cdot\mathbf{w} = 0 \text{ on } \sigma_3 \tag{1.8}$$

For certain values of the parameter p called the critical loads, problem (1.6)-(1.8) for the vector $\mathbf{w}$ can have non-trivial solutions. Let p_0 be one of the eigenvalues of the boundary value problem (1.6)-(1.8). In the general case several eigenfunctions (buckling modes), which we denote by $\boldsymbol{\psi}_m$ $(m = 1, 2, \ldots N)$, can correspond to the eigenvalue p_0.

For $p = p_0$ the boundary value problem (1.3)-(1.5) is solvable not for any right sides $\mathbf{F}, -\mathbf{K}, \boldsymbol{\tau}, \varphi$. To derive the solvability conditions, we multiply the equilibrium Eq. (1.3) by the eigenvector-function $\boldsymbol{\psi}_m$ and integrate over the domain v. Applying the divergence theorem, taking account of the indentity /4/

$$\mathbf{D}^{\cdot}(\nabla\mathbf{w})\cdot\cdot(\nabla\boldsymbol{\psi}_m)^T = \mathbf{D}^{\cdot}(\nabla\boldsymbol{\psi}_m)\cdot\cdot(\nabla\mathbf{w})^T \tag{1.9}$$

and the boundary conditions (1.4), (1.5), (1.7), we obtain

$$\begin{aligned}\int_v \mathbf{k}^{\cdot}(\mathbf{w}, \nabla\mathbf{w})\cdot\boldsymbol{\psi}_m\, dv + \int_{\sigma_1}\mathbf{f}^{\cdot}(\mathbf{w}, \nabla\mathbf{w})\cdot\boldsymbol{\psi}_m\, d\sigma - {}\\ \int_{\sigma_3} S\boldsymbol{\psi}_m\cdot\mathbf{B}\cdot\mathbf{w}\, d\sigma - \int_v \mathbf{D}^{\cdot}(\nabla\boldsymbol{\psi}_m)\cdot\cdot(\nabla\mathbf{w})^T\, dv + {}\\ \int_v \mathbf{K}\cdot\boldsymbol{\psi}_m\, dv + \int_{\sigma_1}\mathbf{F}\cdot\boldsymbol{\psi}_m\, d\sigma + \int_{\sigma_3}\boldsymbol{\tau}\cdot\boldsymbol{\psi}_m\, d\sigma = 0\end{aligned} \tag{1.10}$$

On the basis of (1.4)-(1.7) we have

$$\begin{aligned}\int_v \mathbf{D}^{\cdot}(\nabla\boldsymbol{\psi}_m)\cdot\cdot(\nabla\mathbf{w})^T\, dv = \int_v \mathbf{k}^{\cdot}(\boldsymbol{\psi}_m, \nabla\boldsymbol{\psi}_m)\cdot\mathbf{w}\, dv + {}\\ \int_{\sigma_1}\mathbf{f}^{\cdot}(\boldsymbol{\psi}_m, \nabla\boldsymbol{\psi}_m)\cdot\mathbf{w}\, d\sigma - \int_{\sigma_3} S\mathbf{w}\cdot\mathbf{B}\cdot\boldsymbol{\psi}_m\, d\sigma + \int_{\sigma_3}\mathbf{n}\cdot\mathbf{D}^{\cdot}(\nabla\boldsymbol{\psi}_m)\cdot\mathbf{N}\varphi\, d\sigma\end{aligned} \tag{1.11}$$

Relationship (1.2), expressing the existence of the external force potential, implies the equality /5/

$$\begin{aligned}\int_v \mathbf{k}^{\cdot}(\mathbf{w}, \nabla\mathbf{w})\cdot\boldsymbol{\psi}_m\, dv + \int_{\sigma_1}\mathbf{f}^{\cdot}(\mathbf{w}, \nabla\mathbf{w})\cdot\boldsymbol{\psi}_m\, d\sigma = {}\\ \int_v \mathbf{k}^{\cdot}(\boldsymbol{\psi}_m, \nabla\boldsymbol{\psi}_m)\cdot\mathbf{w}\, dv + \int_{\sigma_1}\mathbf{f}^{\cdot}(\boldsymbol{\psi}_m, \nabla\boldsymbol{\psi}_m)\cdot\mathbf{w}\, d\sigma\end{aligned} \tag{1.12}$$

We obtain from (1.10)-(1.12) and the symmetry property of the tensor $\mathbf{B}$

$$\int_v \mathbf{K}\cdot\boldsymbol{\psi}_m\, dv + \int_{\sigma_1}\mathbf{F}\cdot\boldsymbol{\psi}_m\, d\sigma + \int_{\sigma_3}\boldsymbol{\tau}\cdot\boldsymbol{\psi}_m\, d\sigma - \int_{\sigma_3}\varphi\mathbf{n}\cdot\mathbf{D}^{\cdot}(\nabla\boldsymbol{\psi}_m)\cdot\mathbf{N}\, d\sigma = 0 \tag{1.13}$$

Relationships (1.13), derived as necessary conditions for solvability, are, under certain assumptions formulated below, also the sufficient conditions for solvability of the problem (1.3)-(1.5) for $p = p_0$.

2. We assume that the elastic material satisfies the strict Hadamard inequality /6/

$$\frac{\partial^2 W}{\partial C_{st}\partial C_{mn}} a_m b_n a_s b_t > 0, \quad C_{mn} = \mathbf{i}_m \cdot \mathbf{C} \cdot \mathbf{i}_n$$

where a_m, b_n are components of two arbitrary non-zero vectors **a**, **b**. It follows from this inequality that the system of equations obtained from (1.6) by projections on the coordinate axes $\mathbf{i}_k$ will be strongly elliptic, and homogeneous with an order of homogeneity of two.

We also assume that $\sigma = \sigma_1 \bigcup \sigma_2$, while the boundary conditions (1.7) are supplementary /7/. Since it is difficult to establish supplementarity of the boundary conditions for an elastic material of general form and for an arbitrary fundamental solution, this requirement should be confirmed in each specific problem. We note that the kinematic boundary condition $\mathbf{w} = 0$, which is a Dirichlet type condition, satisfies the supplementarity requirement for any strongly elliptic systems /8/.

Under the requirements stated, the boundary value problem (1.6), (1.7) is elliptic /7/ and corresponding to it is the operator A acting from the Banach space E_1, the vector-functions $\mathbf{w}$ whose components belong to the Sobolev space $W_2^2(v)$ and satisfy the boundary conditions on σ_2, into the Banach space E_2 consisting of the set of functions $h = (-\mathbf{K}, \mathbf{F})$, in which the components of the vector-functions $\mathbf{K}$ belong to the Lebesgue space $L_2(v)$ while the components of the vector-functions $\mathbf{F}$ belong to the Slobodetskii space $W_2^{1/2}(\sigma_1)$. Moroever, by virtue of (1.9), (1.12), the following Green's formula is valid for any vector-functions $\mathbf{w}_1 \in E_1$, $\mathbf{w}_2 \in E_1$

$$\int_v [\nabla \cdot \mathbf{D}^\cdot (\nabla \mathbf{w}_1)] \cdot \mathbf{w}_2 \, dv - \int_{\sigma_1} [\mathbf{n} \cdot \mathbf{D}^\cdot (\nabla \mathbf{w}_1) \cdot \mathbf{w}_2 - \mathbf{f}^\cdot (\mathbf{w}_1, \nabla \mathbf{w}_1) \cdot \mathbf{w}_2] \, d\sigma = \tag{2.1}$$
$$\int_v [\nabla \cdot \mathbf{D}^\cdot (\nabla \mathbf{w}_2)] \cdot \mathbf{w}_1 \, dv - \int_{\sigma_1} [\mathbf{n} \cdot D^\cdot (\nabla \mathbf{w}_2) \cdot \mathbf{w}_1 - \mathbf{f}^\cdot (\mathbf{w}_2, \nabla \mathbf{w}_2) \cdot \mathbf{w}_1] \, d\sigma$$

It follows from (2.1) that problem (1.6), (1.7) is formally selfadjoint. Then according to /7/, relationships (1.13) in the case $\sigma = \sigma_1 \bigcup \sigma_2$ are not only necessary but sufficient for problem (1.3), (1.4) to be solvable under the condition that the right side $h = (-\mathbf{K}, \mathbf{F})$ belong to the space E_2.

We write the boundary value problem (1.3), (1.4) in the operator form

$$A\mathbf{w} = h(\mathbf{w}) \tag{2.2}$$

Here A is a linear operator, and $h(\mathbf{w})$ is a non-linear operator acting from E_1 into E_2. We let E_1^N denote the subspace of zeros of the operator A of dimensionality N with the basis vector-functions $\mathbf{w}_1, \ldots, \mathbf{w}_N$, $E_1^{\infty-N}$ is the supplement of the subspace E_1^N to E_1. Let A^* be the contraction of the operator A in $E_1^{\infty-N}$. Unlike the operator A it will have a bounded inverse operator.

To investigate the equilibrium modes of an elastic body under almost critical loads, we set

$$p = p_0 + \lambda, \quad \mathbf{w} = \mathbf{v} + \mathbf{u}, \quad \mathbf{v} = \sum_{i=1}^{N} \xi_i \boldsymbol{\psi}_i, \quad \mathbf{u} \in E_1^{\infty-N}$$

where λ is a small parameter. The operator Eq.(2.2) takes the form

$$A^*\mathbf{u} = h(\mathbf{v} + \mathbf{u}, \lambda) \tag{2.3}$$

By virtue of the theorem on implicit operators /9/, in a sufficiently small neighbourhood of the point $\mathbf{u} = 0$, $\mathbf{v} = 0$, $\lambda = 0$ there exists a unique solution fo (2.3) $\mathbf{u} = \mathbf{u}(\mathbf{v}, \lambda) = \mathbf{u}(\xi_1, \ldots, \xi_N, \lambda)$, that is continuous and such that $\mathbf{u}(0, 0) = 0$. Substituting this solution into the solvability conditions (1.13), we obtain the bifurcation equations from which the values of the parameters $\xi_1, \ldots, \xi_N$ are determined. A detailed examination of the bifurcation equations is contained in /9/.

3. To illustrate the approach elucidated above for studying the bifurcation of solutions of the static problem we will consider the buckling of a thick hollow cylinder clamped between two rigid fixed plates subjected to an external hydrostatic pressure of intensity q.

When there are no mass forces the equilibrium Eqs.(1.1) have the form

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{\partial}{r\partial\theta} + \mathbf{i}_3 \frac{\partial}{\partial z} \tag{3.1}$$

Here r, θ, z are cylindrical coordinates in the undeformed state of the body, and $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{i}_3$ are their corresponding basis vectors. The boundary conditions of the problem formulated are

$$\mathbf{e}_r \cdot \mathbf{D}|_{r=r_1} = 0, \quad \mathbf{e}_r \cdot \mathbf{D}|_{r=r_0} = -qJ\mathbf{e}_r \cdot (\mathbf{C}^T)^{-1}; \quad J = \det \mathbf{C} \tag{3.2}$$

where r_0, r_1 are the outer and inner radii of the cylinder.

As the governing relationship we take the model of a semilinear material /1/ ($\mathbf{U}$ is the deformation tensor, and μ, ν are constants)

$$\mathbf{D} = 2\mu\left(\frac{\nu s_1}{1-2\nu} - 1\right)\mathbf{U}^{-1}\cdot\mathbf{C} + 2\mu\mathbf{C} \tag{3.3}$$

$$\mathbf{U} = (\mathbf{C}\cdot\mathbf{C}^T)^{1/2}, \quad s_1 = \operatorname{tr}\mathbf{U} - 3$$

The boundary value problem (3.1)-(3.3) has the following axisymmetric solution /1/:

$$\boldsymbol{\rho} = (Q_1\eta + Q_2\eta^{-1})\mathbf{e}_r + \mathbf{i}_3 z, \quad Q_1 = \frac{k-1-pk}{k-1-p(1-2\nu+k)} \tag{3.4}$$

$$Q_2 = \frac{kp}{k-1-p(1-2\nu+k)}, \quad k = \frac{r_1^2}{r_0^2}, \quad p = \frac{q}{2\mu}, \quad \eta = \frac{r}{r_0}$$

We shall seek the plane equilibrium modes close to the solution (3.4), i.e. we set

$$\mathbf{R} = \boldsymbol{\rho} + u(\eta, \theta)\mathbf{e}_r + v(\eta, \theta)\mathbf{e}_\theta \tag{3.5}$$

Taking account of (3.5), we write the boundary value problem (1.3), (1.4) in the form

$$l(x, D)\mathbf{w}(x) = -\mathbf{K}(\mathbf{w}, x) \tag{3.6}$$

$$b(x, D)\mathbf{w}(x) = \mathbf{F}(\mathbf{w}, x) \quad \text{for} \quad \eta = 1, \eta = k_1 \tag{3.7}$$

Here

$$x = (\eta, \theta), \mathbf{w} = (u, v), \mathbf{K} = (K_1, K_2), \mathbf{F} = (F_1, F_2)$$
$$k_1 = r_1 r_0^{-1}, \; l(x, D)\mathbf{w}(x) \equiv [l_{11}(x, D)u(x) + l_{12}(x, D)v(x),$$
$$l_{21}(x, D)u(x) + l_{22}(x, D)v(x)]$$
$$b(x, D)\mathbf{w}(x) \equiv [b_{11}(x, D)u(x) + b_{12}(x, D)v(x),$$
$$b_{21}(x, D)u(x) + b_{22}(x, D)v(x)]$$
$$l_{11}(x, D) = \partial_1^2 + \eta^{-1}\partial_1 + B\eta^{-2}\partial_2^2 - \eta^{-2}$$
$$l_{12}(x, D) = (1 - B)\eta^{-1}\partial_1\partial_2 - (1 + B)\eta^{-2}\partial_2$$
$$l_{21}(x, D) = (1 - B)\eta^{-1}\partial_1\partial_2 + (1 + B)\eta^{-2}\partial_2$$
$$l_{22}(x, D) = \eta^{-2}\partial_2^2 + B\partial_1^2 + B\eta^{-1}\partial_1 - B\eta^{-2}$$
$$b_{11}(x, D) = \nu_1(\partial_1 + \eta^{-1}) - \eta^{-1}p_1$$
$$b_{12}(x, D) = \nu_1\eta^{-1}\partial_2 - \eta^{-1}p_1\partial_2$$
$$b_{21}(x, D) = -\nu_1 B\eta^{-1}\partial_2 + \eta^{-1}p_1\partial_2$$
$$b_{22}(x, D) = \nu_1 B(\partial_1 + \eta^{-1}) - \eta^{-1}p_1$$
$$K_1 = -(1 - \nu)^{-1}(\partial_1 M_1 - \eta^{-1}\partial_2 N_1 + Q_3^{-1}\eta^{-1}\partial_2 N)$$
$$K_2 = -(1 - \nu)^{-1}(\partial_1 N_1 + \eta^{-1}\partial_2 M_1 - Q_3^{-1}\partial_1 N)$$
$$F_1 = (1 - 2\nu)^{-1}(M_1 - 1), \; F_2 = (1 - 2\nu)^{-1}(N_1 - Q_3^{-1}N)$$
$$Q_3 = 2Q_1, \; \nu_1 = (1 - \nu)(1 - 2\nu)^{-1}$$
$$B = (1 - \nu)^{-1}Q_3^{-1}[(1 - \nu)Q_3 - 1]$$
$$\partial_1 = \frac{\partial}{\partial\eta}, \; \partial_2 = \frac{\partial}{\partial\theta}, \; N = \partial_1 v + \eta^{-1}v - \eta^{-1}\partial_2 u$$
$$M = Q_3 + \partial_1 u + \eta^{-1}u + \eta^{-1}\partial_2 v$$
$$N_1 = NQ^{-1}, \; M_1 = MQ^{-1}, \; Q = (M^2 + N^2)^{1/2}$$
$$p_1 = \begin{cases} 1 - p, & \eta = 1 \\ 1 & \eta = k_1 \end{cases}$$

The differential expressions l and b on the left-hand sides of (3.6), (3.7) are linear. The differential expressions $\mathbf{K}$ and $\mathbf{F}$ do not contain linear components. It can be shown that for $\nu \neq 0.5$ system (3.6) is regularly elliptic /7/ while the boundary conditions (3.7) are supplementary. Let us set the linear operator $A\mathbf{w} \equiv (l\mathbf{w}, b\mathbf{w})$, which we define in the Banach space E_1, corresponding to the left-hand sides of (3.6) and (3.7).

We assume that the desired functions u, v belong to the space $W_2^4(v)$, where v: $(k_1 < \eta < 1, 0 \leqslant \theta \leqslant 2\pi)$. Then it can be shown that the right-hand sides of (3.6) and (3.7) belong to E_2. This enables us to write the boundary value problem (3.6), (3.7) in the operator form

$$A\mathbf{w} = \boldsymbol{\tau}, \; \boldsymbol{\tau} \equiv (-\mathbf{K}, \mathbf{F}) \tag{3.8}$$

The necessary and sufficient conditions for the solvability of (3.8) will, in conformity with (1.13), have the form

$$-\int_{k_1}^{1}\int_{0}^{2\pi} \eta\mathbf{K}\cdot\boldsymbol{\psi}_m \, d\eta \, d\theta - \nu_1^{-1}\int_{0}^{2\pi} \eta\mathbf{F}\cdot\boldsymbol{\psi}_m \, d\theta \,\Big|_{\eta=k_1}^{\eta=1} = 0, \quad m = 1, 2, \ldots \tag{3.9}$$

Here $\boldsymbol{\psi}_m = (\psi_m^1, \psi_m^2)$ are eigenvectors of the operator A that form the basis of the subspace of zeros of this operator.

To find the critical values of the parameter p for which bifurcation of the cylinder equilibrium occurs, we consider the linearized problem

$$A\mathbf{w} = 0 \tag{3.10}$$

Following /1/, we seek the eigenfunctions of problem (3.10) in the form

$$u_n = a_n(\eta)\cos n\theta, \ v_n = b_n(\eta)\sin n\theta \tag{3.11}$$
$$b_0(\eta) = 0, \ n = 0, 1, 2, \ldots$$

We substitute (3.11) into the left-hand side of (3.6) and we solve them for a_n, b_n for $K = 0$. We consequently obtain

$$a_0 = \frac{1}{2} A_0\eta + \frac{C_0}{\eta}, \quad b_0 = 0 \tag{3.12}$$
$$a_1 = \frac{3B-1}{8} A_1\eta^2 + \frac{1+B}{2} B_1 \ln\eta + C_1 + \frac{D_1}{\eta^2}$$
$$b_1 = \frac{3-B}{8} A_1\eta^2 + \frac{B-1}{2} B_1 - \frac{1+B}{2} B_1 \ln\eta - C_1 + \frac{D_1}{\eta^2}$$
$$a_n = \frac{2B-Dn}{4(n+1)} A_n\eta^{n+1} - \frac{2B+Dn}{4(n-1)} B_n \frac{1}{\eta^{n-1}} + C_n\eta^{n-1} + D_n \frac{1}{\eta^{n+1}},$$
$$b_n = \frac{2B+D(n+2)}{4(n+1)} A_n\eta^{n+1} - \frac{D(n-2)-2B}{4(n-1)} B_n \frac{1}{\eta^{n-1}} -$$
$$C_n\eta^{n-1} + D_n \frac{1}{\eta^{n+1}}, \quad n = 2, 3, 4, \ldots$$

Here $D = 1 - B$, A_0, C_0, A_i, B_i, C_i, D_i $(i = 1, 2, \ldots)$ are constants of integration.

Substituting (3.11) into the boundary conditions (3.7) with the zero right-hand sides, we take account of (3.12) to obtain a system of equations to determine the constants of integration. For each $n = 2, 3, \ldots$ we find the eigenvalues of the operator A by equating its determinant to zero

$$p = p_0 = \frac{(1-k)S}{2(1-\nu) + (1-k)S}, \quad S = \frac{k-\gamma}{\gamma + k^2 + k\sqrt{(1-k)^2 + 4\gamma}}$$
$$\gamma = \frac{n^2(1-k)^2}{k^n + k^{-n} - 2}, \quad n = 2, 3, \ldots$$

Solving the system obtained for the case $n = 1$ we have $A_1 = B_1 = D_1 = 0$, and C_1 is an arbitrary constant that corresponds to the motion of a cylinder as an absolutely rigid body. Consequently, this case is eliminated from consideration. For $n = 0$ we have $A_0 = B_0 = 0$.

The critical pressure p_* is the minimum eigenvalue from the set of eigennumbers $p_0(n, k)$, where n takes integer values 2, 3, ..., successively, for fixed k. It can be shown that the eigenvalues p_0 are always simple and take on the least value for $n = 2$.

Let λ be a small additional loading parameter. Then setting $p = p_0 + \lambda$ the Eqs.(3.8) can be written in the form (A_0 is the operator A in which the quantity p is replaced by the eigenvalue p_0)

$$A_0\mathbf{w} = \tau - A\mathbf{w} + A_0\mathbf{w} \equiv h(\mathbf{w}), \quad h(\mathbf{w}) \equiv (-\mathbf{t}, \mathbf{T}) \tag{3.13}$$
$$\mathbf{t} = (K^1, K^2), \ \mathbf{T} = (F^1, F^2) \tag{3.14}$$
$$K^1 = K_1 - \eta^{-1}\partial_2 N(A - A_0), \ K^2 = K_2 + (A - A_0)\partial_1 N$$
$$F^1 = F_1 - p_2\eta^{-1}(u + \partial_2 v), \ F^2 = F_2 - \nu_1 N(A - A_0) -$$
$$p_2\eta^{-1}(v - \partial_2 u)$$
$$p_2 = \begin{cases} \lambda, & \eta = 1 \\ 0, & \eta = k_1 \end{cases}$$

We shall seek small solutions of (3.13) in the form of the series

$$\mathbf{w} = \xi\boldsymbol{\psi}_n + \sum_{i=2}^{\infty} \mathbf{w}_{i0}\xi^i + \sum_{i=0}^{\infty} \xi^i \sum_{j=1}^{\infty} \mathbf{w}_{ij}\lambda^j \tag{3.15}$$
$$\mathbf{w}_{ij} \equiv (u_{ij}(\eta, \theta), v_{ij}(\eta, \theta))$$

Here ξ is a formal parameter, $\boldsymbol{\psi}_n = (u_n, v_n)$ is an eigenfunction of the operator A_0. Expanding terms containing u and v in power series in u, v on the right side of (3.13) and the terms containing p in power series in λ, by taking account of (3.15) we obtain

$$h(\mathbf{w}) = \sum_{i+j\geqslant 1} h_{ij}\xi^i\lambda^j, \quad h_{ij} \equiv (-\mathbf{t}_{ij}, \mathbf{T}_{ij}) \tag{3.16}$$
$$\mathbf{t}_{ij} \equiv (K_{ij}^1, K_{ij}^2), \quad \mathbf{T}_{ij} \equiv (F_{ij}^1, F_{ij}^2)$$

where K_{ij}^1, K_{ij}^2, F_{ij}^1, F_{ij}^2 are expansion coefficients of the functions K^1, K^2, F^1, F^2 defined by (3.14).

Substituting (3.15) into (3.13) and equating coefficients of identical powers of $\xi^i\lambda^j$ and taking account of (3.16), we obtain a recurrent system to find $\mathbf{w}_{ij}$

$$A_0{}^*\mathbf{w}_{ij} = h_{ij} \tag{3.17}$$

from which we find all the $\mathbf{w}_{ij}$. Here $A_0{}^*$ is the contraction of the operator A_0 on $E_1^{\infty-1}$.

To obtain the bifurcation equation we substitute (3.15) into the solvability condition (3.9) for (3.13). We consequently obtain

$$\sum_{i=2}^{\infty} L_{i0}\xi^i + \sum_{i=0}^{\infty}\xi^i \sum_{j=1}^{\infty} L_{ij}\lambda^j = 0 \tag{3.18}$$

$$L_{ij} = -\int_{k_1}^{1}\int_{0}^{2\pi} \eta \mathbf{t}_{ij}\cdot\boldsymbol{\psi}_n \, d\eta\, d\theta - \nu_1^{-1}\int_{0}^{2\pi} \eta \mathbf{T}_{ij}\cdot\boldsymbol{\psi}_n \, d\theta \,\Big|_{\eta=k}^{\eta=1}$$

The first coefficients of (3.18) will have the form

$$L_{01} = 0,\ L_{02} = 0,\ L_{20} = 0$$

$$L_{11} = \delta\int_{k_1}^{1}\int_{0}^{2\pi} \eta N_n{}^2\, d\eta\, d\theta + \nu_1^{-1}\int_{0}^{2\pi} (u_n{}^2 + u_n\partial_2 v_n + v_n{}^2 - v_n\partial_2 u_n)\, d\theta|_{\eta=1}$$

$$L_{30} = \delta_1^{-1}\int_{k_1}^{1}\int_{0}^{2\pi} \eta\left(N_n{}^2 M_{20} - Q_{10}^{-1}N_n{}^2 M_n{}^2 + 2N_n M_n N_{20} + \frac{1}{4}Q_{10}^{-1}N_n{}^4\right) d\eta\, d\theta$$

$$\delta = \frac{k-1}{2\nu_1(k-1-p_0k)^2},\quad \delta_1 = 4Q_{10}{}^2(1-\nu),\quad Q_{10} = Q_1|_{p=p_0}$$

$$N_n = \partial_1 v_n + \eta^{-1}v_n - \eta^{-1}\partial_2 u_n,\ M_n = \partial_1 u_n + \eta^{-1}u_n + \eta^{-1}\partial_2 v_n$$

$$N_{20} = \partial_1 v_{20} + \eta^{-1}v_{20} - \eta^{-1}\partial_2 u_{20},\ M_{20} = \partial_1 u_{20} + \eta^{-1}u_{20} + \eta^{-1}\partial_2 v_{20}$$

The bifurcation Eq. (3.18) approximately takes the form $L_{30}\xi^3 + L_{11}\xi\lambda \approx 0$, from which it follows that $\xi = \pm(L_{11}L_{30}{}^{-1}\lambda)^{1/2} + o(\lambda^{1/2})$, and the solution (3.15) of (3.13) is written in the form

$$\mathbf{w} = \pm(-L_{11}L_{30}^{-1}\lambda)^{1/2}\boldsymbol{\psi}_n - (L_{11}L_{30}^{-1})\lambda\mathbf{w}_{20} \pm (-L_{11}L_{30}^{-1}\lambda)^{1/2}\mathbf{w}_{11}\lambda + o(\lambda)$$

Depending on the sign of $(-L_{11}L_{30}{}^{-1})$ two new solutions occur in one of the semicircles $(p_0 - \varepsilon, p_0)$ or $(p_0, p_0 + \varepsilon)$, while new solutions will not occur in the other.

A numerical investigation of the coefficients L_{11}, L_{30} for the case of the minimum eigenvalue p_* showed that always $L_{11} < 0$; $L_{30} > 0$ for $k_1 > 0.5098$; $L_{30} < 0$ for $k_1 < 0.5097$ for all allowable values of ν.

Therefore, for cylinders whose geometrical parameter is characterised by the inequality $k_1 > 0.5098$ (thin-walled cylinders are also included here), equilibrium modes different from the axisymmetric state (3.4) exist for loads greater than the critical p_*. For thick-walled cylinders ($k_1 < 0.5097$), equilibrium modes close to axisymmetric exist only for pressures less than the critical value.

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